2017

MATHEMATICS - PAPER-I

Time Allowed : 3 Hours

Full Marks : 200

If the questions attempted are in excess of the prescribed number, only the questions attempted first up to the prescribed number shall be valued and the remaining ones ignored.

Answers may be given either in English or in Bengali but all answers must be in one and the same language.

1. Answer any two questions :-

(a) In a vector space $R^3(R)$, find a basis of $R^3(R)$ containing the vectors (2, -1, 0) and (1, 3, 2).

(b) A mapping $T : R^3 \to R^3$, defined by $T(x, y, z) = (x + 2y + 3z, 3x + 2y + z, x + y + z)$; $x, y, z \in R^3$. Show that T is linear. Find Ker (T) and dimension of Ker (T).

(c) Apply Gram Schmidt process to obtain an orthogonal basis of R^3 using the standard inner product, given that $\{(1,0,1), (1,0,-1), (0,3,4)\}$ is a basis of R^3.

10 x 2

2. Answer any two questions :-

(a) Using Cauchy's criterian, to prove that

$X_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + (-1)^{n-1} \cdot \frac{1}{n}$ is convergent.

(b) If $y = \cos (m \sin^{-1} x)$, show that -

(i) $(1 - x^2)y_2 - xy_1 + m^2 y = 0$

(ii) $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} + (m^2 - n^2)y = 0$

Where the suffixes of y denote the order of differentiations.

(c) If $y = x^{2n}$ where n is a +ve integer, show that -

$$y_n = 2^n\{1.3.5. \ldots (2n - 1)\} x^n$$

10 x 2

3. Answer any two questions :-

(a) Evaluate $\lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{1/x^2}$

(b) Prove that the asymtotes of the Cubic $(x^2 - y^2)y - 2ay^2 + 5x - 7 = 0$ form a triangle of area 'a^2'.

(c) Find the envelope of the straight line $\frac{x}{a} + \frac{y}{b} = 1$, where 'a' and 'b' are variable parameters, connected by the relation $a+b = c$, c being a non-zero constant.

10 x 2

4. Answer any two questions :-

(a) Show that the function $f : [0, \pi/4] \to R$ defined by

$$f(x) = \begin{cases} \cos x, & \text{when } x \text{ is rational} \\ \sin x, & \text{when } x \text{ is irrational} \end{cases}$$

is not R-integrable on $[0, \pi/4]$

(b) A sequence of functions $\{f_n(x)\}$ is defined on $[0,a]$, $0<a<1$, by $f_n(x) = x^n$, $x \in [0,a]$. Show that the sequence $\{f_n(x)\}$ converges uniformly on $[0, a]$.

P. T. O.

(c) Find the power series of $\tan^{-1}x$ and deduce the sum of

$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$ 10 x 2

5. Answer any two questions :-

(a) The equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of parallel straight lines. Show that the distance between them is $2\sqrt{\frac{g^2 - ac}{a(a + b)}}$

(b) If the two conics $\frac{l_1}{r} = 1 - e_1 \cos\theta$ and $\frac{l_2}{r} = 1 - e_2 \cos(\theta - \alpha)$ touch one another, show that -

$$l_1^2(1 - e_2^2) + l_2^2(1 - e_1^2) = 2l_1l_2(1 - e_1e_2 \cos\alpha).$$

(c) Find the locus of the middle points of the normal chords of the parabola $y^2 = 4ax$. 10 x 2

6. Answer any two questions :-

(a) If '2c' be the shortest distance between the lines $\frac{x}{l} - \frac{z}{n} = 1$, $y = 0$ and $\frac{y}{m} + \frac{z}{n} = 1$, $x = 0$, show that $\frac{1}{c^2} = \frac{1}{l^2} + \frac{1}{m^2} + \frac{1}{n^2}$

(b) A sphere touches the planes $2x + 3y - 6z + 14 = 0$ and $2x + 3y - 6z + 42 = 0$. The centre of the sphere lies on the line $2x + z = 0$, $y = 0$, then find the equation of the sphere.

(c) If '2r' be the distance between two parallel tangent planes to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ which are parallel to the plane $lx + my + nz = 0$ then show that -

$$(a^2 - r^2)l^2 + (b^2 - r^2)m^2 + (c^2 - r^2)n^2 = 0$$ 10 x 2

7. Answer any two questions :-

(a) Reduce the differential equation $y^2 = 2px - y p^2$ to Clairaut's form by the substitution $y^2 = v$ and then obtain its complete primitive and singular solution, if any.

(b) Solve $(D^2 + 2D + 1)y = x \cos x$ where $D \equiv \frac{d}{dx}$

(c) Solve: $x^3 \frac{d^3y}{dx^3} - x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - 2y = x^3$ 10 x 2

8. Answer any two questions :-

(a) Find the Partial Differential Equation (PDE) of all planes which are at a Constant distance 'a' from the origin.

(b) Solve the PDE

$$y^2p - xyq = x(z - 2y)$$

(where p and q have their usual meaning)

(c) Using Laplace transform,

Solve $(D^2 + 2D + 1)y = 3te^{-z}$,

given that $y = 4$ and $Dy = 2$ when $t = 0$ 10 x 2

Contd...P/3.

9. Answer any two questions :-

(a) A beam whose centre of gravity divides it into two portions, 'a' and 'b', is placed inside a smooth sphere. Show that if θ be its inclination to the horizon in the position of equilibrium and 2α be the angle subtended by the beam at the centre of the sphere then $\tan\theta = \frac{b - a}{b + a}\tan\alpha$.

(b) A uniform chain of length 'l' is to be suspended from two points A and B in the same horizontal line so that either terminal tension is n-times that of the lowest point. Show that the span AB must be

$$\frac{l}{\sqrt{n^2 - 1}} \log_e \left(n + \sqrt{n^2 - 1}\right)$$

(c) A regular hexagon is composed of six equal heavy rods freely jointed together and two opposite angles are connected by a string which is horizontal. One rod being in contact with a horizontal plane. At the middle point of the opposite rod a weight w_1 is placed. If w be the weight of each rod, show that the tension of the string is $\frac{3w + w_1}{\sqrt{3}}$. 10 x 2

10. Answer any two questions :-

(a) A particle moves in a straight line, its accelaration is directed towards a fixed point 'O' in the line and is always equal to $\mu(a^5/x^2)^{1/3}$, when it is at a distance 'x' from O. If it starts from rest at a distance 'a' from O, show that it will arrive at O with velocity $a\sqrt{6\mu}$ after time $\frac{8}{15}\sqrt{6/\mu}$.

(b) A light string passes over a light fixed pulley, it carries a mass 'P' at one extrimity and a light pulley at the other. Another light string passes over this second pulley, carring masses 'R' and 'Q' at its extrimities. The system starts from rest. If R always remains at rest,

Prove that $\frac{4}{P} + \frac{1}{Q} = \frac{3}{R}$

(c) The velocities of a particle along and perpendicular to a radius vector from a fixed origin are λr^2 and $\mu\theta^2$, where 'λ' and 'μ' are constants. Show that the equation of the path is $\frac{\mu}{2r^2} = \frac{\lambda}{\theta} + c$ where 'c' is an arbitrary constant and the components of acceleration are $2\lambda^2 r^3 - \frac{\mu^2\theta^4}{r}$ and $\lambda\mu r\theta^2 + \frac{2\mu^2\theta^3}{r}$ 10 x 2