

2021

MATHEMATICS

PAPER-I

Time Allowed — 3 Hours *Full Marks — 200*

If the questions attempted are in excess of the prescribed number, only the questions attempted first up to the prescribed number shall be valued and the remaining ones ignored.

Answers may be given either in **English** *or in* **Bengali** *but all answers must be in one and the same language.*

1. Answer *any two* questions: 10×2=20

(a) Consider the following subspaces of $\mathbb{R}^5$:

$U = \text{span}\{(1, 3, -2, 2, 3), (1, 4, -3, 4, 2), (2, 3, -1, -2, 9)\}$

$W = \text{span}\{(1, 3, 0, 2, 1), (1, 5, -6, 6, 3), (2, 5, 3, 2, 1)\}$

Find a basis and the dimension of U+W. Hence find the dimension of $U \cap W$.

(b) Let $f: \mathbb{R}^4 \to \mathbb{R}^3$ be the linear mapping defined by

$f(x, y, z, \omega) = (x-y+z+\omega, x+2z-\omega, x+y+3z-3\omega)$

Find a basis and the dimension of the (i) image of f and (ii) the kernel of f.

(c) Using Cayley-Hamilton Theorem, find A^9 where

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

2. Answer *any two* questions: 10×2=20

(a) Prove that the sequence $\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}$, ... is monotonic increasing. Is the sequence bounded? Justify your answer.

(b) Find the value of $\lim_{x \to 0}\left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right)$.

(c) Show that the maximum value of $\left(\frac{1}{x}\right)^x$ is $e^{1/e}$.

3. Answer *any two* questions: 10×2=20

(a) Evaluate $\lim_{n\to\infty}\left\{\frac{n!}{(pn)^n}\right\}^{1/n}$, $p \neq 0$.

(b) Test the convergence of the series $2-\frac{3}{2}+\frac{4}{3}-\frac{5}{4}+.....$

(c) Evaluate, if possible $\int_0^{\pi}\frac{dx}{1-\cos x}$.

4. Answer *any two* questions: 10×2=20

(a) In the Mean Value Theorem, $f(h)=f(o)+hf'(\theta h), 0<\theta<1$. Show that the limiting value of θ as $h\to 0$ is $\frac{1}{2}$ where $f(x)=\cos x$.

(b) Find all the asymptotes of the curve $y=e^{6x}$.

(c) If $f(x)=x^n$, prove that

$$f(1)+\frac{f'(1)}{1!}+\frac{f''(1)}{2!}+\frac{f'''(1)}{3!}+.....+\frac{f^n(1)}{n!}=2^n.$$

5. Answer *any two* questions: 10×2=20

(a) Find the equation of the pair of straight lines through the origin and perpendicular to the pair of straight lines given by $3x^2+xy-2y^2-5x-5y=0$.

(b) If by a transformation of rotation of co-ordinate axes, the expression $cx+dy$ is changed into $c'x'+d'y'$, where c, c', d, d' are constants, then show that $c'^2+d'^2=c^2+d^2$.

(c) If the normal to an ellipse at the point P meets the major and minor axes at G and H respectively, then show that

PG.PH=SP.S'P

where S, S' are focii of the ellipse.

6. Answer *any two* questions: 10×2=20

(a) A variable plane which is at a constant distance $3p$ from the origin intersects the co-ordinate axes at A, B, C. Show that the locus of the centroid of Δ ABC is

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{p^2}.$$

(b) Find the length of the shortest distance between the lines

$$\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1} \text{ and } \frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}.$$

(c) Find the equation of the sphere which passes through the points (1,0,0), (0,1,0) and (0,0,1) and radius as small as possible.

7. Answer *any two* questions: 10×2=20

(a) Reducing the differential equation $x^2p^2+p(2x+y)+y^2=0$ to Clairaut's form by the substitution $y=u,\ xy=v$, solve it. Find singular solution/s.

(b) Solve : $(D^2+D+1)y=e^{2x}$

(c) Solve the equation by the method of variation of parameters

$$\frac{d^2y}{dx^2}+a^2y=\sec\ ax$$

8. Answer *any two* questions: 10×2=20

(a) Solve : $z^2-pz+qz+(x+y)^2=0$

(b) Using Charpit's method, solve the equation $(p^2+q^2)y=qz$.

(c) Evaluate $L\{F(t)\}$ where

$$F(t)=\begin{cases}(t-1)^2, & t>1\\ 0, & 0<t<1\end{cases}$$

9. Answer *any two* questions: 10×2=20

(a) Two uniform similar rods of same material PQ and QT of length $2a$ and $2b$ respectively are rigidly united at Q and suspended freely from P. If they rest inclined at angles α and β respectively to the vertical, prove that $(a^2+2ab)\sin\alpha=b^2\sin\beta$.

(b) A solid hemisphere of weight W rests in limiting equlibrium with its curved surface on a rough inclined plane and the plane face is horizontal by a weight P attached at a point in the rim. Prove that the co-efficient of friction is $\frac{P}{\sqrt{W(2P+W)}}$.

(c) Four equal rods each of weight W form a rhombus ABCD with smooth hinges at the joints. The frame is suspended by the end A and a weight W' is attached at C. A stiffering rod of negligible weight joins the middle points of AB and AD keeping these inclined at an angle α to AC. Show that the thrust in the stiffening rod is $(4W + 2W')\tan\alpha$.

10. Answer *any two* questions: 10×2=20

(a) The velocities of a point parallel to the axes of x and y are $u+\omega y$ and $v+\omega' x$ respecitvely, where u, v, ω, ω' are constants. Show that the path of the point is a conic.

(b) A particle moving with a simple harmonic motion in a straight line has velocities v_1, v_2 at distances x_1, x_2 from the centre of its path. Show that, if T be the period of its motion, then,

$$T = 2\pi\sqrt{\frac{x_1^2 - x_2^2}{v_2^2 - v_1^2}}.$$

(c) Assuming the moon to describe a circular orbit of radius 4×10^5 km round the earth in 27·3 days, calculate the periodic time of an artificial satellite of the earth near the earth's surface (Radius of the earth = 6400 km).
