Group-B

Answer *any two* questions.

7. (a) Consider simple random sampling with replacement SRSWR of size n from a population of size N. Find the first and second order inclusion probabilities.

(b) Show that the sample mean $\bar{y}$ based on an SRSWR of size n can be expressed as $\bar{y} = \sum_1^N t_i Y_i / n$, where $(t_1, t_2, \ldots, t_N)$ follows a multinomial distribution. Hence find $E(\bar{y})$ and $\text{Var}(\bar{y})$.

(c) Let $\bar{y}'$ be the sample mean based on distinct units of an SRSWR of size n. Show that $E(\bar{y}') = \bar{Y}$, $\text{Var}(\bar{y}') \le \text{Var}(\bar{y})$ where $\bar{Y}$ and $\bar{y}$ have usual meaning. 10+15+15=40

8. (a) Consider the model $E(y_{ij}) = \mu + \alpha_i + \beta_j, i = 1, 2, \ldots, p; j = 1, 2, \ldots, q;$ $\sum_{i=1}^{p} \alpha_i = 0 = \sum_{j=1}^{q} \beta_j$.

Define $\hat{\alpha}_i = y_{i0} - y_{00}, \hat{\beta}_j = y_{0j} - y_{00}, \hat{\mu} = y_{00}$

where $y_{i0} = \sum_{j=1}^{q} y_{ij} / q, y_{0j} = \sum_{i=1}^{p} y_{ij} / p, y_{00} = \sum_i \sum_j y_{ij} / pq$.

Prove the following identity

$$\sum_i \sum_j (y_{ij} - \mu - \alpha_i - \beta_j)^2 = SS_e + q \sum_i (\hat{\alpha}_i - \alpha_i)^2 + p \sum_j (\hat{\beta}_j - \beta_j)^2 + pq(\hat{\mu} - \mu)^2$$

where $SS_e = \sum_i \sum_j (y_{ij} - y_{i0} - y_{0j} + y_{00})^2$. Use the identity to find

(i) the least squares estimates of μ, α_i and β_j,

(ii) sum of squares (s.s) due to $H_{01} : \alpha_i = 0 \ \forall i$; s.s due to $H_{02} : \beta_j = 0 \ \forall j$ and

(iii) to prove the partition of s.s

$$\sum_i \sum_j (y_{ij} - y_{00})^2 = q \sum_i (y_{i0} - y_{00})^2 + p \sum_j (y_{0j} - y_{00})^2 + SS_e.$$

Construct the tests for H_{01} and H_{02} with usual assumptions on $\{y_{ij}\}$.

(b) Consider the one-way classification model $E(y_{ij}) = t_i, j = 1, 2, \ldots, n_i, i = 1, 2, \ldots, k$,

where $\text{Cov}(y_{ij}, y_{i'j'}) = 0$ if $i \ne i'$

$= 0$ if $j \ne j'$ when $i = i'$

$= \sigma^2$ if $i = i', j = j'$.

Assume that $\sum_1^k n_i = n = k.p$ plots are available for experimentation. Determine n_is so that $\sum_{i=1}^{k} \text{Var}(\hat{t}_i)$ is minimum, where $\hat{t}_i$ is the least squares estimator of t_i (p is a positive integer). 20+20=40

9. (a) Suppose 5 fodders are to be tested for increasing yield of milk. 25 cows are available which are divided into 5 age groups and 5 breeds. Describe a statistical design so that the fodders can be effectively tested. Give the layout of the design without randomisation. Also give the analysis of the observations arising out of the design.

(b) Construct a confounded balanced $(2^3, 2)$ plan where all the interactions are balanced. (Use minimum number of replications and give the confounding plan only.)

(c) Describe the Yates' method of computing the factorial effects (including the total effect) in a 2^2 factorial experiment from the treatment effects. Show that Yates' method is based on the algebraic relation

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

20+10+10=40

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(c) Let the p.d.f. of the joint distribution of $X_1, X_2, \ldots, X_p$ be

$$\frac{1}{\left(\sqrt{2\pi}\right)^p} e^{-\frac{1}{2}\Sigma x_i^2} \left\{ 1 + \prod_{i=1}^{p} x_i \, e^{-\frac{1}{2}\sum_1^p \left(x_i^2 - 1\right)} \right\}; \ -\infty < x_i < \infty \ \forall i.$$

Find the marginal distribution of $X_i, i = 1, 2, \ldots, p$. 8+14+8=30

4. (a) State and prove the Fisher-Neyman factorization theorem in the discrete set-up.

(b) Show that the largest order statistic in a random sample $(X_1, X_2, \ldots, X_n)$ from a distribution with pmf $P(X = k) = \frac{1}{N}, k = 1, 2, \ldots, N$, N being a positive integer, is sufficient for N. 16+14=30

5. (a) Let U be the class of all unbiased estimators of θ with $E_\theta\left(T^2\right) < \infty$ for all θ and suppose U is non-empty. Let U_0 be the class of all unbiased estimator T_0 of 0, i.e.

$$\mathsf{U}_0 = \left\{ T_0 : E\left(T_0\right) = 0, \, E_\theta\left(T_0^2\right) < \infty \ \forall \ \theta \right\}.$$

Then prove that T is UMVUE if and only if $E_\theta\left(T \, T_0\right) = 0$ for all θ and all $T_0 \in \mathsf{U}_0$.

(b) Let $(X_1, X_2, \ldots, X_n)$ form a random sample from the rectangular distribution with pdf $f_\theta(x) = \frac{1}{\theta}, 0 < x < \theta; 0 < \theta < \infty$. Show that $Y = \frac{n+1}{n} X_{(n)}$ is the UMVUE of θ with $\mathrm{Var}(Y) = \theta^2 / n(n+2)$ which is less than the Cramer-Rao lower bound, ($X_{(n)}$ is the largest order statistic). 15+15=30

6. (a) In connection with testing statistical hypotheses, define (i) uniformly most powerful (ii) unbiased (iii) uniformly most powerful unbiased critical regions.

(b) Let $X \sim$ Normal (0.1) under H_0 and $X \sim$ Cauchy with pdf $f(x) = \frac{1}{\left(1 + x^2\right)}$, $-\infty < x < \infty$. Find an MP test for H_0 against H_1.

(c) For testing $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$, show that there exists a non-negative constant k such that a critical region W_0 defined by

$$W_0 = \left\{ \left(x_1, \ldots, x_n\right) / \, f_{\theta_1}\left(x_1, \ldots, x_n\right) / \, f_{\theta_0}\left(x_1, \ldots, x_n\right) > k \right\}$$

satisfies $\int_{W_0} f_\theta\left(x_1, \ldots, x_n\right) dx_1, \ldots, dx_n = \infty$

Also show that W_0 is an MP critical region for testing H_0 against H_1. (Assume that $f_{\theta_1}\left(X_1, X_2, \ldots, X_n\right) / f_{\theta_0}\left(X_1, X_2, \ldots, X_n\right)$ is continuous.) 6+12+12=30

2021

STATISTICS

PAPER-I

Time Allowed — 3 Hours *Full Marks — 200*

If the questions attempted are in excess of the prescribed number, only the questions attempted first up to the prescribed number shall be valued and the remaining ones ignored.

Answers may be given either in ***English*** *or in* ***Bengali*** *but all answers must be in one and same language.*

Group-A

Answer *any four* questions.

1. (a) Give the axiomatic definition of probability. Describe how the limitations of classical definition of probability are taken care of. Also show that the classical definition of probability can also be obtained from this definition in particular case.

(b) State and prove Poincaré's theorem on probability of occurrence of at least one of the events $A_1, A_2, \ldots\ldots, A_n$.

(c) A player tosses an unbiased coin n times on the condition that he gains Re. 1 if he casts 'Head'; otherwise he looses Re. 1. Find the probability he neither looses nor gains. 7+15+8=30

2. (a) Define a random variable and its cumulative distribution function (CDF). Show that a CDF is non-decreasing. If F is a CDF then show that $G(x)$ defined by $G(x)=\frac{1}{2h}\int_{x-h}^{x+h}F(y)dy$ is also a CDF.

(b) Prove the following recursion relation among the central moments of $N(\mu,\sigma^2)$, (a normal distribution with mean μ and variance σ^2)

$$\mu_{r+1}=\sigma^3\frac{d\mu_r}{d\sigma}+\sigma^2\mu_r. \quad r=0, 1, 2, \ldots.$$

Hence find the first four central moments. 18+12=30

3. (a) Let $\{X_n\}, n=1, 2, \ldots$ be a sequence of random variables. When is $\{X_n\}$ said to obey weak law of large numbers (WLLN)? Let $(X_1, X_2, \ldots, X_n)$ form a random sample from a Bernoulli distribution. Show that the sample proportion of successes converges in probability to the population proportion.

(b) Let X be an absolutely continuous random variable. Show that

$$\sum_{n=1}^{\infty}\text{Prob}\left(|X|\geq n\right)\leq E|X|\leq 1+\sum_{n=1}^{\infty}\text{Prob}\left(|X|\geq n\right).$$