2022

MATHEMATICS

PAPER-I

Time Allowed — 3 Hours *Full Marks — 200*

If the questions attempted are in excess of the prescribed number, only the questions attempted first up to the prescribed number shall be valued and the remaining ones ignored.

*Answers may be given either in **English** or in **Bengali** but all answers must be in one and same language.*

1. Answer *any two* questions. 10×2=20

   (a) Show the mapping $T : V_2(R) \to V_3(R)$ defined as $T(a, b) = (a + b, a - b, b)$ is a linear transformation from $V_2(R)$ into $V_3(R)$. Find the range, rank, null, space and nullity of T. 4+2+2+2=10

   (b) If λ be an eigenvalue of an orthogonal matrix, then show that $\frac{1}{\lambda}$ is also an eigenvalue.

   (c) If a_1, a_2, a_3 be fixed elements of a field F, then show that the set W of all ordered triads (x_1, x_2, x_3) of elements of F such that $a_1x_1 + a_2x_2 + a_3x_3 = 0$ is a sub-space of V_3 in F.

2. Answer *any two* questions: 10×2=20

   (a) Prove that the sequence $\{x_n\}$ whose n-th term is $x_n = \sqrt{(n+1)} - \sqrt{n}$ converges and find its limit. 8+2=10

   (b) Find the envelop of the family of co-axial ellipses $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where the parameters a and b are connected by $a^n + b^n = c^n$.

   (c) Find the value of $\lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{\frac{1}{x}}$.

3. Answer *any two* questions: 10×2=20

   (a) Using definition of compact set prove that the set [0, 1] is not a compact set in R.

   (b) Show that $\int_0^1 \log\left(\frac{1+x}{1-x}\right) \cdot \frac{x^3}{\sqrt{1-x^2}} dx$ is convergent.

   (c) The function f is defined by the equality $f(x) = 1 + 2 \cdot 4x + 3 \cdot 4^2x^2 + 4 \cdot 4^3x^3 + \ldots + \cdot n \cdot 4^{n-1} x^{n-1} + \ldots$

   Show that f is continuous on $\left(-\frac{1}{4}, \frac{1}{4}\right)$. Evaluate $\int_0^{\frac{1}{8}} f(x)dx$. 7+3=10

4. Answer *any two* questions: 10×2=20

(a) Show that $\sum_{n=1}^{\infty} xe^{-nx}$ is not uniformly convergent on [0, 1].

(b) Find the asymptotes of the curve $x=\frac{t^2}{1+t^3}$, $y=\frac{t^2+2}{1+t}$.

(c) If $y=\cos\left(10\cos^{-1}x\right)$, show that $\left(1-x^2\right)y_{12}=21xy_{11}$.

5. Answer *any two* questions: 10×2=20

(a) Find the condition that the line $\frac{1}{r}=A\cos\theta+B\sin\theta$, may touch the conic $\frac{l}{r}=1+e\cos\theta$.

(b) Show that if one of the lines given by $ax^2+2hxy+by^2=0$ be perpendicular to one of the lines given by $a'x^2+2h'xy+b'y^2=0$, then $(aa'-bb')^2+4(ah'+hb')(ha'+bh')=0$.

(c) If the perpendicular straight lines $ax+by+c=0$ and $bx-ay+c'=0$ be taken as the axes of x and y respectively, then show that the equation $(ax+by+c)^2-2(bx-ay+c')^2=1$ will be transformed into $y'^2-2x'^2=\frac{1}{a^2+b^2}$.

6. Answer *any two* questions: 10×2=20

(a) Obtain the equation of the plane containing the line $\frac{x}{a}+\frac{z}{c}=1$, $y=0$ and parallel to the line $\frac{y}{b}-\frac{z}{c}=1$, $x=0$.

(b) The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, meets the coordinate axes in A, B and C. Prove that the equation of the cone generated by lines drawn from the origin to meet the circle ABC is $yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0$.

(c) Show that the line $\frac{x+2}{2}=\frac{y}{3}=\frac{z-1}{-2}$ is a generator of the quadric $\frac{x^2}{4}-\frac{y^2}{9}=z$.

7. Answer *any two* questions: 10×2=20

(a) Find orthogonal trajectories of $r^n\sin nx=a^n$.

(b) Solve: $(D^2+1)\,y=3\cos^2x+2\sin^3x$

(c) Find the singular solution of $y^2\left(y-x\frac{dy}{dx}\right)=x^4\left(\frac{dy}{dx}\right)^2$.

8. Answer *any two* questions: 10×2=20

(a) Solve the boundary value problem $y''+2y'+y=0$, given $y(0)=0$ and $y(1)=2$, where $y''=\frac{d^2y}{dx^2}$ and $y'=\frac{dy}{dx}$ by using Laplace transform.

(b) Apply Charpit's method to solve the differential equation $pxy+pq+qy-yz=0$.

(c) Solve $(t+y+z)\frac{\partial t}{\partial x}+(t+z+x)\frac{\partial t}{\partial y}+(t+x+y)\frac{\partial t}{\partial x}=x+y+z$ by Lagrang's method.

9. Answer *any two* questions: 10×2=20

(a) A square, of side $2a$, is placed with its plane vertical between two smooth pegs, which in same horizontal line and at a distance c. Show that it will be in equilibrium when the inclination of one of its edges to the horizon is either 45° or $\frac{1}{2}\sin^{-1}\left(\frac{a^2-c^2}{c^2}\right)$.

(b) A ladder whose c.g. divides it into two portion of length 'a' and 'b' rest with one end on a horizontal floor and other end against a rough vertical wall. If the coefficient of friction at the floor and the wall respectively μ and μ', show that the inclination of the ladder to the floor, when the equilibrium is limiting, is $\tan^{-1}\frac{a-b\mu\mu'}{\mu(a+b)}$.

(c) Two forces act, one along the line $y = 0, z = 0$ and the other along the line $x = 0, z = c$. As the forces vary, show that the surface generated by the axis of their equivalent wrench is $(x^2 + y^2)\, z = cy^2$.

10. Answer *any two* questions: 10×2=20

(a) Find the law of force to the pole when the path is the cardioid $r = a\,(1 - \cos\theta)$.

(b) If v_1 and v_2 are the linear velocities of a planet when it is respectively nearest and farthest from the sun, prove that $(1 - e)\, v_1 = (1 + e)\, v_2$.

(c) An engine is pulling a train and works at a constant power doing H units of work per second. If M be the mass of the whole train and F the resistance supposed to be constant, show that the time generating the velocity v from rest is $\left(\frac{MH}{F^2}\log\frac{H}{H-Fv} - \frac{Mv}{F}\right)$ seconds.