MWC(O)-MATH-I/23

2023
MATHEMATICS
PAPER-I

Time Allowed — 3 Hours   *Full Marks — 200*

If the questions attempted are in excess of the prescribed number, only the questions attempted first up to the prescribed number shall be valued and the remaining ones ignored.

*Answers may be given either in **English** or in **Bengali** but all answers must be in one and same language.*

1. *Answer* any two *questions:*   10×2=20

   (a) If $U = L\{(1, 2, 1), (2, 1, 3)\}$, $W = L\{(1, 0, 0), (0, 0, 1)\}$, show that U and W are subspaces of $\mathbb{R}^3$. Determine dim U, dim W, dim $(U \cap W)$. Deduce that dim $(U + W) = 3$.   2+2+4+2

   (b) Determine the linear mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$, which maps the basis vectors (0, 1, 1), (1, 0, 1), (1, 1, 0) of $\mathbb{R}^3$ to (1, 1, 1), (1, 1, 1), (1, 1, 1) respectively. Verify that dim (Ker T) + dim (ImT) = 3.   5+5

   (c) Diagonalize the symmetric matrix A, where $A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$.   10

2. *Answer* any two *questions:*   10×2=20

   (a) Prove that the sequence $\{u_n\}$ defined by $u_1 = \sqrt{7}$ and $u_{n+1} = \sqrt{7 + u_n}$ for all $n \geq 1$ converges to the positive root of the equation $x^2 - x - 7 = 0$ 10

   (b) Find a and b in order that $\lim_{x \to 0} \dfrac{a \sin 2x - b \sin x}{x^3} = 1$.   10

   (c) Show that the envelope of circles whose centres lie on the rectangular hyperbola $xy = c^2$ and which passes through its centre is $\left(x^2 + y^2\right)^2 = 16c^2 xy$.   10

3. *Answer* any two *questions:*   **WBCS MADE EASY**   10×2=20

   (a) Using the definition of a compact set, prove that a finite subset of $\mathbb{R}$ is a compact set in $\mathbb{R}$.   10

   (b) Prove that the integral $\int_a^b \dfrac{dx}{(x-a)^\mu}$ is convergent if and only if $\mu < 1$.   10

   (c) Test the convergence of the series

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots$$

    10

4. *Answer* any two *questions:* 10×2=20

(a) Prove that $\dfrac{\pi^2}{9} < \displaystyle\int_{\pi/6}^{\pi/2} \dfrac{x}{\sin x}\,dx < \dfrac{2\pi^2}{9}$. 10

(b) Find the asymptotes of $x^3 + x^2 y - xy^2 - y^3 + x^2 - y^2 = 2$. 10

(c) Let $f_n(x) = \dfrac{x}{n + x^2}, x \in [0,1]$. Show that the sequence $\{f_n\}$ is uniformly convergent on $[0,1]$. 10

5. *Answer* any two *questions:* 10×2=20

(a) Reduce the equation $3x^2 + 2xy + 3y^2 - 18x - 22y + 50 = 0$ to its standard form by applying a rotation $\dfrac{\pi}{4}$ which is followed by a translation about the point (2, 3). 10

(b) Find the product of the perpendiculars from (x_1, y_1) to the straight lines represented by $ax^2 + 2hxy + by^2 = 0$. 10

(c) Show that the locus of the middle points of normal chords of the rectangular hyperbola $x^2 - y^2 = a^2$ is $(y^2 - x^2)^3 = 4a^2 x^2 y^2$. 10

6. *Answer* any two *questions:* 10×2=20

(a) A variable plane at a constant distance p from the origin meets the axes at A, B, C. Show that the locus of the centroid of the tetrahedron $OABC$ is $\dfrac{1}{x^2} + \dfrac{1}{y^2} + \dfrac{1}{z^2} = \dfrac{16}{p^2}$. 10

(b) Find the equations of the tangent planes to the conicoid $2x^2 - 6y^2 + 3z^2 = 5$ which pass through the straight line $x + 9y - 3z = 0 = 3x - 3y + 6z - 5$. 10

(c) Find the equations to the generating lines of the hyperboloid $\dfrac{x^2}{4} + \dfrac{y^2}{9} - \dfrac{z^2}{16} = 1$, which pass through the point (2, 3, − 4). 10

7. *Answer* any two *questions:* 10×2=20

(a) Find the orthogonal trajectories of cardioids $r = a(1 - \cos\theta), a$ being a parameter. 10

(b) Solve : $(D^2 - 2D + 1)y = x^2 e^x$ 10

(c) Solve the Euler – Cauchy equation : $(x^2 D^2 - xD + 2)y = x \log x$ 10

8. *Answer* any two *questions:*   10×2=20

    (a) Solve : $(x^2 - y^2 - z^2)p + 2xyq = 2xz$ 10

    (b) Using Charpit's method find a complete integral of the *PDE* $px + qy = pq$.   10

    (c) Solve $(D^2 + 6D + 9)y = \sin t$ using Laplace transform, where $y(0) = 1, y^1(0) = 0$.   10

9. *Answer* any two *questions:*   10×2=20

    (a) A square frame *ABCD* of four equal joined rods hangs from *A*, the shape being maintained by a string joining mid-points of *AB, BC*. Prove that the ratio of the tension of the string to the reaction at *C* is $\dfrac{8}{\sqrt{5}}$.   10

    (b) A perfectly rough plane is inclined at an angle α to the horizon. Show that the least eccentricity of the ellipse which can rest on the plane is $\sqrt{\dfrac{2\sin\alpha}{1+\sin\alpha}}$.   10

    (c) A force p acts along the axis of x and another force np acts along a generator of the cylinder $x^2 + y^2 = a^2$; show that the central axis lies on the cylinder $n^2(nx - z)^2 + (1 + n^2)^2 y^2 = n^4 a^2$.

                                                                10

10. *Answer* any two *questions:*   10×2=20

    (a) A mass m when suspended from a light spring causes an extension α. If a mass M is added to m, find the periodic time of the ensuing oscillation together with the amplitude of the oscillation.

                                                                 10

    (b) If the orbit described by a particle under a central force to the origin be $r^n \cos n\theta = a^n$, find the law of force.   10

    (c) A body is projected horizontally from a point on the earth's surface with the velocity $\sqrt{1\cdot5Rg}$, R being the earth's radius. What will be its maximum distance from the earth's centre?   10

———————